COMMON FIXED POINT THEOREMS FOR TWO MAPPINGS IN $M$-FUZZY METRIC SPACES
SHABAN SEDGHI*, JUNG HWA IM, AND NABI SHOBE

Abstract

In this paper, we prove some common fixed point theorems for two nonlinear mappings in complete $M$-fuzzy metric spaces. Our main results improved versions of several fixed point theorems in complete fuzzy metric spaces.

1. Introduction

The concept of fuzzy sets was introduced initially by Zadeh [27] in 1965. Since then, to apply this concept in topology and analysis, many authors [9, 17, 19, 24] have expansively developed the theory of fuzzy sets and application. George and Veeramani [8] and Kramosil and Michalek [11] have introduced the concept of fuzzy topological spaces induced by fuzzy metric which have very important applications in quantum particle physics particularly in connections with both string and $E$-infinity theory which were given and studied by El-Naschie [3-6]. Many authors [7, 10, 12, 18, 20, 23] have proved fixed point theorem in fuzzy (probabilistic) metric spaces. Vasuki [25] obtained the fuzzy version of common fixed point theorem which had extra conditions. In fact, Vasuki [25] proved fuzzy common fixed point theorem by a strong definition of a Cauchy sequence (see Note 3.13 and Definition 3.15 of [8], also [23, 26]).

On the other hand, Dhage [1, 2] introduced the notion of generalized metric or $D$-metric spaces and claimed that $D$-metric convergence defines a Hausdorff topology and $D$-metric is sequentially continuous in all the three variables. Many authors have used these claims in proving fixed point theorems in $D$-metric spaces, but, unfortunately, almost all theorems in $D$-metric spaces are not valid (see [13-16, 22]).

Recently, Sedghi et al. [21] introduced $D^*$-metric which is a probable modification of the definition of $D$-metric introduced by Dhage [1, 2] and proved
some basic properties in $D^*$-metric spaces. Also, using the concept of the $D^*$-metrics, they defined $M$-fuzzy metric space and proved some related fixed point theorems for some nonlinear mappings in complete $M$-fuzzy metric spaces.

In this paper, we prove some common fixed point theorems for two nonlinear mappings in complete $M$-fuzzy metric spaces. Our main results improved versions of several fixed point theorems in complete fuzzy metric spaces.

In what follows $(X, D^*)$ will denote a $D^*$-metric space, $N$ the set of all natural numbers and $R^+$ the set of all positive real numbers.

**Definition 1.1.** ([21]) Let $X$ be a nonempty set. A generalized metric (or $D^*$-metric) on $X$ is a function: $D^*: X^3 \rightarrow R^+$ that satisfies the following conditions: for any $x, y, z, a \in X$,

1. $D^*(x, y, z) \geq 0$,
2. $D^*(x, y, z) = 0$ if and only if $x = y = z$,
3. $D^*(x, y, z) = D^*(p\{x, y, z\})$ (symmetry), where $p$ is a permutation function,
4. $D^*(x, y, z) \leq D^*(x, y, a) + D^*(a, z, z)$.

The pair $(X, D^*)$ is called a generalized metric space (or $D^*$-metric space).

Some immediate examples of such a function are as follows:

(a) $D^*(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\}$.
(b) $D^*(x, y, z) = d(x, y) + d(y, z) + d(z, x)$, where $d$ is the ordinary metric on $X$.
(c) If $X = R^n$, then we define $D^*(x, y, z) = (||x - y||^p + ||y - z||^p + ||z - x||^p)^{1/p}$ for any $p \in R^+$.
(d) If $X = R^+$, then we define $D^*(x, y, z) = \begin{cases} 0, & \text{if } x = y = z, \\ \max\{x, y, z\}, & \text{otherwise}. \end{cases}$

In a $D^*$-metric space $(X, D^*)$, we can prove that $D^*(x, x, y) = D^*(x, y, y)$.

Let $(X, D^*)$ be a $D^*$-metric space. For any $r > 0$, define the open ball with the center $x$ and radius $r$ as follows:

$B_{D^*}(x, r) = \{y \in X : D^*(x, y, y) < r\}$.

**Example 1.2.** Let $X = R$. Denote $D^*(x, y, z) = |x - y| + |y - z| + |z - x|$ for all $x, y, z \in R$. Thus we have $B_{D^*}(1, 2) = \{y \in R : D^*(1, y, y) < 2\} = \{y \in R : |y - 1| + |y - 1| < 2\} = \{y \in R : |y - 1| < 1\} = (0, 2)$.

**Definition 1.3.** ([21]) Let $(X, D^*)$ be a $D^*$-metric space and $A \subset X$.
(1) If, for any \( x \in A \), there exists \( r > 0 \) such that \( B_{D^*}(x, r) \subset A \), then \( A \) is called an open subset of \( X \).

(2) A subset \( A \) of \( X \) is said to be \( D^*\)-bounded if there exists \( r > 0 \) such that \( D^*(x, y, z) < r \) for all \( x, y \in A \).

(3) A sequence \( \{x_n\} \) in \( X \) is said to be convergent to a point \( x \in X \) if
\[
D^*(x_n, x_n, x) = D^*(x, x, x_n) \rightarrow 0 \quad (n \rightarrow \infty).
\]
That is, for any \( \epsilon > 0 \), there exists \( n_0 \in N \) such that
\[
D^*(x, x, x_n) < \epsilon, \quad \forall n \geq n_0.
\]
Equivalently, for any \( \epsilon > 0 \), there exists \( n_0 \in N \) such that
\[
D^*(x, x_n, x_m) < \epsilon, \quad \forall n, m \geq n_0.
\]

(4) A sequence \( \{x_n\} \) in \( X \) is called a Cauchy sequence if, for any \( \epsilon > 0 \), there exits \( n_0 \in N \) such that
\[
D^*(x_n, x_n, x_m) < \epsilon, \quad \forall n, m \geq n_0.
\]

(5) A \( D^*\)-metric space \((X, D^*)\) is said to be complete if every Cauchy sequence is convergent.

Let \( \tau \) be the set of all \( A \subset X \) with \( x \in A \) if and only if there exists \( r > 0 \) such that \( B_{D^*}(x, r) \subset A \). Then \( \tau \) is a topology on \( X \) (induced by the \( D^*\)-metric \( D^* \)).

**Definition 1.4.** ([21]) Let \((X, D^*)\) be a \( D^*\)-metric space. \( D^* \) is said to be a continuous function on \( X^3 \times (0, \infty) \) if
\[
\lim_{n \to \infty} D^*(x_n, y_n, z_n) = D^*(x, y, z)
\]
whenever a sequence \( \{(x_n, y_n, z_n)\} \) in \( X^3 \) converges to a point \( (x, y, z) \in X^3 \), i.e.,
\[
\lim_{n \to \infty} x_n = x, \quad \lim_{n \to \infty} y_n = y, \quad \lim_{n \to \infty} z_n = z.
\]

**Remark 1.5.** ([21]) (1) Let \((X, D^*)\) be a \( D^*\)-metric space. Then \( D^* \) is continuous function on \( X^3 \).

(2) If a sequence \( \{x_n\} \) in \( X \) converges to a point \( x \in X \), then the limit \( x \) is unique.

(3) If a sequence \( \{x_n\} \) in \( X \) is converges to a point \( x \), then \( \{x_n\} \) is a Cauchy sequence in \( X \).

Recently, motivated by the concept of \( D^*\)-metrics, Sedghi et al. [21] introduced the concept of \( M \)-fuzzy metric spaces and their properties and, further, proved some related common fixed theorems for some contractive type mappings in \( M \)-fuzzy metric spaces.

**Definition 1.6.** ([21]) A binary operation \( \ast : [0, 1] \times [0, 1] \rightarrow [0, 1] \) is a continuous \( t \)-norm if it satisfies the following conditions:

(1) \( \ast \) is associative and commutative,
Two typical examples of continuous $t$-norm are $a \ast b = ab$ and $a \ast b = \min(a, b)$.

**Definition 1.7.** ([21]) A 3-tuple $(X, \mathcal{M}, \ast)$ is called an $\mathcal{M}$-fuzzy metric space if $X$ is an arbitrary (non-empty) set, $\ast$ is a continuous $t$-norm and $\mathcal{M}$ is a fuzzy set on $X^3 \times (0, \infty)$ satisfying the following conditions: for all $x, y, z, a \in X$ and $t, s > 0$,

1. $\mathcal{M}(x, y, z, t) > 0$,
2. $\mathcal{M}(x, y, z, t) = 1$ if and only if $x = y = z$,
3. $\mathcal{M}(x, y, z, t) = \mathcal{M}(p\{x, y, z\}, t)$ (symmetry), where $p$ is a permutation function,
4. $\mathcal{M}(x, y, a, t) \ast \mathcal{M}(a, z, z, s) \leq \mathcal{M}(x, y, z, t + s),$
5. $\mathcal{M}(x, y, z, t) : X^3 \times (0, \infty) \to [0, 1]$ is continuous with respect to $t$.

**Remark 1.8.** Let $(X, \mathcal{M}, \ast)$ be an $\mathcal{M}$-fuzzy metric space. Then, for any $t > 0$, $\mathcal{M}(x, x, y, t) = \mathcal{M}(x, y, y, t)$.

Let $(X, \mathcal{M}, \ast)$ be an $\mathcal{M}$-fuzzy metric space. For any $t > 0$, the open ball $B_\mathcal{M}(x, r, t)$ with the center $x \in X$ and radius $0 < r < 1$ is defined by

$$B_\mathcal{M}(x, r, t) = \{y \in X : \mathcal{M}(x, y, y, t) > 1 - r\}.$$

A subset $A$ of $X$ is called an open set if, for all $x \in A$, there exist $t > 0$ and $0 < r < 1$ such that $B_\mathcal{M}(x, r, t) \subseteq A$.

**Definition 1.9.** ([21]) Let $(X, \mathcal{M}, \ast)$ be an $\mathcal{M}$-fuzzy metric space.

1. A sequence $\{x_n\}$ in $X$ is said to be convergent to a point $x \in X$ if $\mathcal{M}(x, x, x_n, t) \to 1$ as $n \to \infty$ for any $t > 0$.
2. A sequence $\{x_n\}$ is called a Cauchy sequence if, for any $0 < \epsilon < 1$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that
   $$\mathcal{M}(x_n, x_m, x_n, t) > 1 - \epsilon, \quad \forall n, m \geq n_0.$$
3. An $\mathcal{M}$-fuzzy metric space $(X, \mathcal{M}, \ast)$ is said to be complete if every Cauchy sequence in $X$ is convergent.

**Example 1.10.** Let $X$ is a nonempty set and $D^\ast$ be the $D^\ast$-metric on $X$. Denote $a \ast b = a \cdot b$ for all $a, b \in [0, 1]$. For any $t \in [0, \infty]$, define

$$\mathcal{M}(x, y, z, t) = \frac{t}{t + D^\ast(x, y, z)}, \quad \forall x, y, z \in X.$$ 

It is easy to see that $(X, \mathcal{M}, \ast)$ is a $\mathcal{M}$-fuzzy metric space.

**Remark 1.11.** Let $(X, \mathcal{M}, \ast)$ be a fuzzy metric space. If we define $\mathcal{M} : X^3 \times (0, \infty) \to [0, 1]$ by

$$\mathcal{M}(x, y, z, t) = \mathcal{M}(x, y, t) \ast \mathcal{M}(y, z, t) \ast \mathcal{M}(z, x, t), \quad \forall x, y, z \in X,$$

then $(X, \mathcal{M}, \ast)$ is an $\mathcal{M}$-fuzzy metric space.
Lemma 1.12. ([21]) Let $(X, M_*, *)$ be an $M$-fuzzy metric space. Then, for all $x, y, z \in X$ and $t > 0$, $M(x, y, z, t)$ is nondecreasing with respect to $t$.

Definition 1.13. ([21]) Let $(X, M_*, *)$ be an $M$-fuzzy metric space. $M$ is said to be continuous function on $X^3 \times (0, \infty)$ if
\[
\lim_{n \to \infty} M(x_n, y_n, z_n, t_n) = M(x, y, z, t)
\]
whenever a sequence $\{(x_n, y_n, z_n, t_n)\}$ in $X^3 \times (0, \infty)$ converges to a point $(x, y, z, t) \in X^3 \times (0, \infty)$, that is,
\[
\lim_{n \to \infty} x_n = x, \quad \lim_{n \to \infty} y_n = y, \quad \lim_{n \to \infty} z_n = z,
\]
\[
\lim_{n \to \infty} M(x, y, z, t_n) = M(x, y, z, t).
\]

Lemma 1.14. ([21]) Let $(X, M_*, *)$ be an $M$-fuzzy metric space. Then $M$ is continuous function on $X^3 \times (0, \infty)$.

Lemma 1.15. ([21]) Let $(X, M_*, *)$ be an $M$-fuzzy metric space. If we define $E_{\lambda, M}: X^3 \to R^+ \cup \{0\}$ by
\[
E_{\lambda, M}(x, y, z) = \inf\{t > 0 : M(x, y, z, t) > 1 - \lambda\}, \quad \forall \lambda \in (0, 1),
\]
then we have the following:
1. For any $\mu \in (0, 1)$, there exists $\lambda \in (0, 1)$ such that
\[
E_{\mu, M}(x_1, x_2, x_n) \leq E_{\lambda, M}(x_1, x_2) + E_{\lambda, M}(x_2, x_3) + \cdots + E_{\lambda, M}(x_{n-1}, x_n)
\]
for any $x_1, x_2, \cdots, x_n \in X$.
2. A sequence $\{x_n\}$ is convergent in an $M$-fuzzy metric space $(X, M_*, *)$ if and only if $E_{\lambda, M}(x_0, x_n, x) \to 0$. Also, the sequence $\{x_n\}$ is a Cauchy sequence in $X$ if and only if it is a Cauchy sequence with $E_{\lambda, M}$.

Lemma 1.16. ([21]) Let $(X, M_*, *)$ be an $M$-fuzzy metric space. If there exists $k > 1$ such that
\[
M(x_n, x_n, x_{n+1}, t) \geq M(x_0, x_0, x_1, k^n t), \quad \forall n \geq 1,
\]
then $\{x_n\}$ is a Cauchy sequence in $X$.

Definition 1.17. ([7]) We say that an $M$-fuzzy metric space $(X, M_*, *)$ has the property (C) if it satisfies the following condition: For some $x, y, z \in X$,
\[
M(x, y, z, t) = C, \quad \forall t > 0, \quad \Rightarrow \quad C = 1.
\]
2. The main results

Now, we are ready to give main results in this paper.

**Theorem 2.1.** Let \((X, \mathcal{M}, \ast)\) be a complete \(\mathcal{M}\)-fuzzy metric space and \(S, T\) be two self-mappings of \(X\) satisfying the following conditions:

(i) there exists a constant \(k \in (0, 1)\) such that

\[
\mathcal{M}(Sx, TX, Ty, kt) \geq \gamma(\mathcal{M}(x, Sx, y, t)), \quad \forall x, y \in X, \tag{2.1}
\]

or

\[
\mathcal{M}(Ty, STy, Sx, kt) \geq \gamma(\mathcal{M}(y, Ty, x, t)), \quad \forall x, y \in X, \tag{2.2}
\]

where \(\gamma : [0, 1] \to [0, 1]\) is a function such that \(\gamma(a) \geq a\) for all \(a \in [0, 1]\),

(ii) \(ST = TS\).

If \((X, \mathcal{M}, \ast)\) have the property \((C)\), then \(S\) and \(T\) have a unique common fixed point in \(X\).

**Proof.** Let \(x_0\) be an arbitrary point in \(X\), define

\[
\begin{cases}
  x_{2n+1} = Tx_{2n}, \\
  x_{2n+2} = Sx_{2n+1},
\end{cases} \quad \forall n \geq 0. \tag{2.3}
\]

(1) Let \(d_m(t) = \mathcal{M}(x_m, x_{m+1}, x_{m+1}, t)\) for any \(t > 0\). Then, for any even \(m = 2n \in \mathbb{N}\), by (2.1) and (2.3), we have

\[
d_{2n}(kt) = \mathcal{M}(x_{2n}, x_{2n+1}, x_{2n+1}, kt)
\]

\[
= \mathcal{M}(Sx_{2n-1}, Tx_{2n}, Tx_{2n}, kt)
\]

\[
= \mathcal{M}(Sx_{2n-1}, TSx_{2n-1}, T_{2n}, kt)
\]

\[
\geq \gamma(\mathcal{M}(x_{2n-1}, Sx_{2n-1}, x_{2n}, t))
\]

\[
\geq \mathcal{M}(x_{2n-1}, x_{2n}, x_{2n}, t)
\]

\[
= d_{2n-1}(t).
\]

Thus \(d_{2n}(kt) \geq d_{2n-1}(t)\) for all even \(m = 2n \in \mathbb{N}\) and \(t > 0\).

Similarly, for any odd \(m = 2n + 1 \in \mathbb{N}\), we have also

\[
d_{2n+1}(kt) \geq d_{2n}(t).
\]

Hence we have

\[
d_n(kt) \geq d_{n-1}(t), \quad \forall n \geq 1. \tag{2.4}
\]

Thus, by (2.4), we have

\[
\mathcal{M}(x_n, x_{n+1}, x_{n+1}, t) \geq \mathcal{M}(x_{n-1}, x_n, x_n, \frac{1}{k}t) \geq \cdots \geq \mathcal{M}(x_0, x_1, x_1, \frac{1}{k^n}t).
\]

Therefore, by Lemma 1.16, \(\{x_n\}\) is a Cauchy sequence in \(X\) and, by the completeness of \(X\), \(\{x_n\}\) converges to a point \(x\) in \(X\) and so

\[
\lim_{n \to \infty} x_{2n+1} = \lim_{n \to \infty} Tx_{2n} = \lim_{n \to \infty} Sx_{2n+1} = \lim_{n \to \infty} x_{2n+2} = x.
\]
Now, we prove that $Tx = x$. Replacing $x, y$ by $x_{2n-1}, x$, respectively, in (i), we obtain
\[ M(Sx_{2n-1}, TSx_{2n-1}, Tx, kt) \geq \gamma(M(x_{2n-1}, Sx_{2n-1}, x, t)), \]
that is,
\[ M(x_{2n}, x_{2n+1}, Tx, kt) \geq \gamma(M(x_{2n-1}, x_{2n}, x, t)) \geq M(x_{2n-1}, x_{2n}, x, t). \] (2.5)

Letting $n \to \infty$ in (2.5), we have
\[ M(x, x, Tx, kt) \geq M(x, x, x, t) = 1, \]
which implies that $Tx = x$, that is, $x$ is a fixed point of $T$.

Next, we prove that $Sx = x$. Replacing $x, y$ by $x, x_{2n}$, respectively, in (2.1), we obtain
\[ M(Sx, TSx, Tx_{2n}, kt) \geq \gamma(M(x, Sx, x_{2n}, t)) \geq M(x, Sx, x_{2n}, t). \]

By (ii), since $TS = ST$, we get
\[ M(Sx, Sx, Tx_{2n}, kt) \geq \gamma(M(x, Sx, x_{2n}, t)) \geq M(x, Sx, x_{2n}, t). \] (2.6)

Letting $n \to \infty$ in (2.6), we have
\[ M(Sx, Sx, x, kt) \geq M(x, Sx, x, t) \]
and hence
\[ M(x, Sx, x, t) \geq M(x, Sx, x, \frac{1}{k} t) \]
\[ \geq M(x, Sx, x, \frac{1}{k^2} t) \]
\[ \ldots \]
\[ \geq M(x, Sx, x, \frac{1}{k^n} t). \]

On the other hand, it follows from Lemma 1.12 that
\[ M(x, Sx, x, k^n t) \leq M(x, Sx, x, t). \]

Hence $M(x, Sx, x, t) = C$ for all $t > 0$. Since $(X, M, \ast)$ has the property (C), it follows that $C = 1$ and so $Sx = x$, that is, $x$ is a fixed point of $S$. Therefore, $x$ is a common fixed point of the self-mappings $S$ and $T$. 

(2) By using (2.2) and (2.3), let 
\[ d_m(t) = \mathcal{M}(x_{m+1}, x_m, x_m, t) \]
for any \( t > 0 \). Then, for any even \( m = 2n \in \mathbb{N} \), we have
\[ d_{2n}(kt) = \mathcal{M}(x_{2n+1}, x_{2n}, x_{2n}, kt) \]
\[ = \mathcal{M}(Tx_{2n}, Sx_{2n-1}, Sx_{2n-1}, kt) \]
\[ = \mathcal{M}(Tx_{2n}, STx_{2n-2}, Sx_{2n-1}, kt) \]
\[ \geq \gamma(\mathcal{M}(x_{2n}, Tx_{2n-2}, x_{2n-1}, t)) \]
\[ \geq \mathcal{M}(x_{2n}, Tx_{2n-2}, x_{2n-1}, t) \]
\[ = d_{2n-1}(t). \]

Thus \( d_{2n}(kt) \geq d_{2n-1}(t) \) for all even \( m = 2n \in \mathbb{N} \) and \( t > 0 \).
Similarly, for any odd \( m = 2n + 1 \in \mathbb{N} \), we have also
\[ d_{2n+1}(kt) \geq d_{2n}(t). \]

Hence we have
\[ d_n(kt) \geq d_{n-1}(t), \quad \forall n \geq 1. \]

The remains of the proof are almost same to the case of (2.1).

Now, to prove the uniqueness, let \( x' \) be another common fixed point of \( S \) and \( T \). Then we have
\[ \mathcal{M}(x, x, x', kt) = \mathcal{M}(Sx, TSx, Tx', kt) \]
\[ \geq \gamma(\mathcal{M}(x, Sx, x', t)) \]
\[ \geq \mathcal{M}(x, x, x', t), \]
which implies that
\[ \mathcal{M}(x, x, x', t) \geq \mathcal{M}(x, x, x', \frac{1}{k}t) \]
\[ \geq \mathcal{M}(x, x, x', \frac{1}{k^2}t) \]
\[ \geq \mathcal{M}(x, x, x', \frac{1}{k^n}t). \]

On the other hand, it follows from Lemma 2.12 that
\[ \mathcal{M}(x, x, x', t) \leq \mathcal{M}(x, x, x', \frac{1}{k^n}t) \]
and hence \( \mathcal{M}(x, x, x', t) = C \) for all \( t > 0 \). Since \((X, \mathcal{M}, \ast)\) has the property \((C)\), it follows that \( C = 1 \), that is, \( x = x' \). Therefore, \( x \) is a unique common fixed point of \( S \) and \( T \). This completes the proof. \( \square \)

By Theorem 2.1, we have the following:
Corollary 2.2. Let \((X, M, *)\) be a complete \(M\)-fuzzy metric space. Let \(T\) be a mapping from \(X\) into itself such that there exists a constant \(k \in (0, 1)\) such that
\[
M(Tx, T^2x, Ty, kt) \geq M(x, Tx, y, t), \quad \forall x, y \in X.
\]
If \((X, M, *)\) have the property \((C)\), then \(T\) have a unique fixed point in \(X\).

**Proof.** By Theorem 2.1, if we set \(\gamma(a) = a\) and \(S = T\), then the conclusion follows. \(\square\)

Corollary 2.3. Let \((X, M, *)\) be a complete \(M\)-fuzzy metric space. Let \(T\) be a mapping from \(X\) into itself such that there exists a constant \(k \in (0, 1)\) such that
\[
M(T^n x, T^{2n}x, T^n y, kt) \geq M(x, T^n x, y, t)
\]
for all \(x, y \in X\) and \(n \geq 2\). If \((X, M, *)\) has the property \((C)\), then \(T\) have a unique fixed point in \(X\).

**Proof.** By Corollary 2.2, \(T^n\) have a unique fixed point in \(X\). Thus there exists \(x \in X\) such that \(T^n x = x\). Since
\[
T^{n+1} x = T^n(Tx) = T(T^n x) = Tx,
\]
we have \(Tx = x\). \(\square\)

Next, by using Lemma 1.16 and the property \((C)\), we can prove the main results in this paper.

**Theorem 2.4.** Let \((X, M, *)\) be a complete \(M\)-fuzzy metric space with \(t * t = t\) for all \(t \in [0, 1]\). Let \(S\) and \(T\) be mappings from \(X\) into itself such that there exists a constant \(k \in (0, 1)\) such that
\[
M(Sx, Ty, Ty, kt) 
\] 
\[ 
\geq a(t)M(x, Sx, Sx, t) + b(t)M(y, Ty, Ty, t) 
\] 
\[ 
+ c(t)M(x, Ty, Ty, \alpha t) + d(t)M(y, Sx, Sx, (2 - \alpha) t) 
\] 
\[ 
+ e(t)M(x, y, y, t) 
\]
for all \(x, y \in X\) and \(\alpha \in (0, 2)\), where \(a, b, c, d, e : [0, \infty) \to [0, 1]\) are five functions such that
\[
a(t) + b(t) + c(t) + d(t) + e(t) = 1, \quad \forall t \in [0, \infty).
\]

Then \(S\) and \(T\) have a unique common fixed point in \(X\).

**Proof.** Let \(x_0 \in X\) be an arbitrary point. Then there exist \(x_1, x_2 \in X\) such that \(x_1 = Sx_0\) and \(x_2 = Tx_1\). Inductively, we can construct a sequence \(\{x_n\}\) in \(X\) such that
\[
\begin{cases} 
  x_{2n+1} = Sx_{2n}, \\
  x_{2n+2} = Tx_{2n+1}, \quad \forall n \geq 0.
\end{cases} 
\]

Now, we show that \(\{x_n\}\) is a Cauchy sequence in \(X\). If we set
\[
d_m(t) = M(x_m, x_{m+1}, x_{m+1}, t), \quad \forall t > 0,
\]
then \(\{x_n\}\) is a Cauchy sequence in \(X\). If we set
\[
d_m(t) = M(x_m, x_{m+1}, x_{m+1}, t), \quad \forall t > 0,
\]
then we prove that \( \{d_m(t)\} \) is increasing with respect to \( m \in N \). In fact, for any odd \( m = 2n + 1 \in N \), we have

\[
\begin{align*}
    d_{2n+1}(kt) &= \mathcal{M}(x_{2n+1}, x_{2n+2}, x_{2n+2}, x_{2n+1}, k t) \\
    &= \mathcal{M}(S x_{2n}, T x_{2n+1}, T x_{2n+1}, k t) \\
    &\geq a(t)\mathcal{M}(x_{2n}, S x_{2n}, S x_{2n}, t) + b(t)\mathcal{M}(x_{2n+1}, T x_{2n+1}, T x_{2n+1}, t) \\
    &+ c(t)\mathcal{M}(x_{2n}, T x_{2n+1}, T x_{2n+1}, \alpha t) \\
    &+ d(t)\mathcal{M}(x_{2n+1}, S x_{2n}, S x_{2n}, (2 - \alpha) t) \\
    &+ e(t)\mathcal{M}(x_{2n}, x_{2n+1}, x_{2n+1}, t) \\
    &= a(t)\mathcal{M}(x_{2n}, x_{2n+1}, x_{2n+1}, t) + b(t)\mathcal{M}(x_{2n+1}, x_{2n+2}, x_{2n+2}, t) \\
    &+ c(t)\mathcal{M}(x_{2n}, x_{2n+2}, x_{2n+2}, \alpha t) \\
    &+ d(t)\mathcal{M}(x_{2n+1}, x_{2n+1}, x_{2n+1}, (2 - \alpha) t) \\
    &+ e(t)\mathcal{M}(x_{2n}, x_{2n+1}, x_{2n+1}, t)
\end{align*}
\]

and so

\[
\begin{align*}
    d_{2n+1}(kt) &\geq a(t)d_{2n}(t) + b(t)d_{2n+1}(t) + c(t)d_{2n}(t) * d_{2n+1}(qt) \\
    &+ d(t) + e(t)d_{2n}(t).
\end{align*}
\]

The equality in (2.10) is true because, if set \( \alpha = 1 + q \) for any \( q \in (k, 1) \), then

\[
\begin{align*}
    \mathcal{M}(x_{2n}, x_{2n+2}, x_{2n+2}, (1 + q)t) &= \mathcal{M}(x_{2n}, x_{2n}, x_{2n+2}, (1 + q)t) \\
    &\geq \mathcal{M}(x_{2n}, x_{2n}, x_{2n+1}, t) * \mathcal{M}(x_{2n+1}, x_{2n+2}, x_{2n+2}, qt) \\
    &= d_{2n}(t) * d_{2n+1}(qt).
\end{align*}
\]

Now, we claim that

\[
    d_{2n+1}(t) \geq d_{2n}(t), \quad \forall n \geq 1.
\]

In fact, if \( d_{2n+1}(t) < d_{2n}(t) \), then since

\[
    d_{2n+1}(qt) * d_{2n}(t) \geq d_{2n+1}(qt) * d_{2n+1}(qt) = d_{2n+1}(qt)
\]

in (3.10), we have

\[
\begin{align*}
    d_{2n+1}(kt) &> a(t)d_{2n+1}(qt) + b(t)d_{2n+1}(qt) + c(t)d_{2n+1}(qt) \\
    &+ d(t)d_{2n+1}(qt) + e(t)d_{2n+1}(qt) \\
    &= d_{2n+1}(qt)
\end{align*}
\]
and so $d_{2n+1}(kt) > d_{2n+1}(qt)$, which is a contradiction. Hence $d_{2n+1}(t) \geq d_{2n}(t)$ for all $n \in \mathbb{N}$ and $t > 0$. By (2.10), we have
\[
d_{2n+1}(kt) \geq a(t)d_{2n}(qt) + b(t)d_{2n}(qt) + c(t)d_{2n}(qt) + d(t)d_{2n}(qt) + e(t)d_{2n}(qt)
\]
\[
= d_{2n}(qt).
\]
Now, if $m = 2n$, then, by (2.9), we have
\[
d_{2n}(kt)
= \mathcal{M}(x_{2n}, x_{2n+1}, x_{2n+1}, kt)
= \mathcal{M}(Sx_{2n-1}, Tx_{2n}, Tx_{2n}, kt)

\geq a(t)\mathcal{M}(x_{2n-1}, Sx_{2n-1}, Sx_{2n-1}, t) + b(t)\mathcal{M}(x_{2n}, Tx_{2n}, Tx_{2n}, t)
+ c(t)\mathcal{M}(x_{2n-1}, Tx_{2n}, Tx_{2n}, \alpha t)
+ d(t)\mathcal{M}(x_{2n}, Tx_{2n}, Sx_{2n-1}, (2 - \alpha)t)
+ e(t)\mathcal{M}(x_{2n-1}, x_{2n}, x_{2n}, t)

= a(t)\mathcal{M}(x_{2n-1}, x_{2n}, x_{2n}, t) + b(t)\mathcal{M}(x_{2n}, x_{2n+1}, x_{2n+1}, t)
+ c(t)\mathcal{M}(x_{2n-1}, x_{2n+1}, x_{2n+1}, \alpha t) + d(t)\mathcal{M}(x_{2n}, x_{2n}, x_{2n}, (2 - \alpha)t)
+ e(t)\mathcal{M}(x_{2n-1}, x_{2n}, x_{2n}, t)
\]
and so
\[
d_{2n}(kt) \geq a(t)d_{2n-1}(t) + b(t)d_{2n}(t) + c(t)d_{2n-1}(t) * d_{2n}(qt)
+ d(t) + e(t)d_{2n-1}(t).
\] (2.11)
The equality in (2.11) is true because, if $\alpha = 1 + q$ for any $q \in (k, 1)$, then
\[
\mathcal{M}(x_{2n-1}, x_{2n+1}, x_{2n+1}, (1 + q)t)
= \mathcal{M}(x_{2n-1}, x_{2n-1}, x_{2n+1}, (1 + q)t)
\geq \mathcal{M}(x_{2n-1}, x_{2n-1}, x_{2n}, t) * \mathcal{M}(x_{2n}, x_{2n+1}, x_{2n+1}, qt)
\]
\[
= d_{2n-1}(t) * d_{2n}(qt).
\]
Now, we also claim that
\[
d_{2n}(t) \geq d_{2n-1}(t), \quad \forall n \geq 1.
\]
In fact, if $d_{2n}(t) < d_{2n-1}(t)$, then, since
\[
d_{2n}(qt) * d_{2n-1}(t) \geq d_{2n}(qt) * d_{2n}(qt) = d_{2n}(qt)
\] in (3.11), we have
\[
d_{2n}(kt)
> a(t)d_{2n}(qt) + b(t)d_{2n}(qt) + c(t)d_{2n}(qt) + d(t)d_{2n}(qt) + e(t)d_{2n}(qt)
= d_{2n}(qt)
and so \(d_{2n}(kt) > d_{2n}(qt)\), which is a contradiction. Hence \(d_{2n}(t) \geq d_{2n-1}(t)\) for all \(n \in N\) and \(t > 0\). By (2.11), we have

\[
d_{2n}(kt) \\
\geq a(t)d_{2n-1}(qt) + b(t)d_{2n-1}(qt) + c(t)d_{2n-1}(qt) + d_{2n-1}(qt) \\
\]

and so \(d_{2n}(kt) \geq d_{2n-1}(qt)\). Thus we have

\[
d_n(kt) \geq d_{n-1}(qt), \quad \forall n \geq 1.
\]

Therefore, it follows that

\[
M(x_n, x_{n+1}, x_{n+1}, t) \geq M(x_{n-1}, x_n, x_n, t) \geq \cdots \geq M(x_0, x_1, (\frac{q}{k})^n t).
\]

Hence, by Lemma 1.16, \(\{x_n\}\) is a Cauchy sequence in \(X\) and, by the completeness of \(X\), \(\{x_n\}\) converges to a point \(x \in X\) and

\[
\lim_{n \to \infty} x_{2n+1} = \lim_{n \to \infty} Sx_{2n} = \lim_{n \to \infty} Tx_{2n+1} = \lim_{n \to \infty} x_{2n+2} = x.
\]

Now, we prove that \(Sx = x\). In fact, letting \(\alpha = 1\), \(x = x\) and \(y = x_{2n+1}\) in (2.7), respectively, we obtain

\[
M(Sx, Tx_{2n+1}, Tx_{2n+1}, kt) \\
\geq a(t)M(x, Sx, Sx, t) + b(t)M(x_{2n+1}, Tx_{2n+1}, Tx_{2n+1}, t) \\
+ c(t)M(x, Tx_{2n+1}, Tx_{2n+1}, t) + d(t)M(x_{2n+1}, Sx, Sx, t) \\
+ e(t)M(x, x_{2n+1}, x_{2n+1}, t).
\]

If \(Sx \neq x\), then, letting \(n \to \infty\) in (2.12), we have

\[
M(Sx, x, x, kt) \\
\geq a(t)M(x, Sx, Sx, t) + b(t)M(x, x, x, t) \\
+ c(t)M(x, x, x, t) + d(t)M(x, Sx, Sx, t) + e(t)M(x, x, x, t) \\
> M(x, x, Sx, t),
\]

which is a contradiction. Thus it follows that \(Sx = x\).

Similarly, we can prove that \(Tx = x\). In fact, again, replacing \(x\) by \(x_{2n}\) and \(y\) by \(x\) in (2.7), respectively, for \(\alpha = 1\), we have

\[
M(Sx_{2n}, Tx, Tx, kt) \\
\geq a(t)M(x_{2n}, Sx_{2n}, Sx_{2n}, t) + b(t)M(x, Tx, Tx, t) \\
+ c(t)M(x_{2n}, Tx, Tx, t) + d(t)M(x, Sx_{2n}, Sx_{2n}, t) \\
+ e(t)M(x_{2n}, x, x, t)
\]
and so, if $Tx \neq x$, letting $n \to \infty$ in (2.13), we have
\[
\mathcal{M}(x, Tx, Tx, kt) \\
\geq a(t)\mathcal{M}(x, x, x, t) + b(t)\mathcal{M}(x, Tx, Tx, t) \\
+ c(t)\mathcal{M}(x, Tx, Tx, t) + d(t)\mathcal{M}(x, x, x, t) + e(t)\mathcal{M}(x, x, x, t) \\
> \mathcal{M}(x, Tx, Tx, t),
\]
which implies that $Tx = x$. Therefore, $Sx = Tx = x$ and $x$ is a common fixed point of the self-mappings $S$ and $T$ of $X$.

The uniqueness of a common fixed point $x$ is easily verified by using the hypothesis. In fact, if $x'$ be another fixed point of $S$ and $T$, then, for $\alpha = 1$, by (2.7), we have
\[
\mathcal{M}(x, x', x', kt) \\
= \mathcal{M}(Sx, Tx', Tx', kt) \\
\geq a(t)\mathcal{M}(x, Sx, Sx, t) + b(t)\mathcal{M}(x', Tx', Tx', t) \\
+ c(t)\mathcal{M}(x, Tx', Tx', t) + d(t)\mathcal{M}(x', Sx, Sx, t) + e(t)\mathcal{M}(x, x', x', t) \\
> \mathcal{M}(x, x', x', t).
\]
and so $x = x'$.

Example 2.5. Let $(X, \mathcal{M}, *)$ be an $\mathcal{M}$-fuzzy metric space, where $X = [0, 1]$ with $t$-norm defined $a * b = \min\{a, b\}$ for all $a, b \in [0, 1]$ and
\[
\mathcal{M}(x, y, z, t) = \frac{t}{t + |x - y| + |y - z| + |x - z|}, \quad \forall t > 0, x, y, z \in X.
\]
Define the self-mappings $T$ and $S$ on $X$ as follows:
\[
Tx = 1, \quad Sx = \begin{cases} 
1 & \text{if } x \text{ is rational,} \\
0 & \text{if } x \text{ is irrational.}
\end{cases}
\]
We can find the functions $a, b, c, d, e : [0, \infty) \to [0, 1]$ such that $a(t) + b(t) + c(t) + d(t) + e(t) = 1$ and the following inequality holds:
\[
\mathcal{M}(Sx, Ty, Ty, kt) \\
\geq a(t)\mathcal{M}(x, Sx, Sx, t) + b(t)\mathcal{M}(y, Ty, Ty, t) \\
+ c(t)\mathcal{M}(x, Ty, Ty, \alpha t) + d(t)\mathcal{M}(y, Sx, Sx, (2 - \alpha)t) \\
+ e(t)\mathcal{M}(x, y, y, t).
\]
It is easy to see that the all the conditions of Theorem 3.4 hold and 1 is a unique common fixed point of $S$ and $T$.

From Theorem 2.4, we have the following:
Corollary 2.6. Let \((X, M, \ast)\) be a complete \(M\)-fuzzy metric space with \(t \ast t = t\) for all \(t \in [0, 1]\). Let \(S\) be a mapping from \(X\) into itself such that there exists \(k \in (0,1)\) such that
\[
M(Sx, Sy, Sy, kt) \\
\geq a(t)M(x, Sx, Sx, t) + b(t)M(y, Sy, Sy, t) \\
+ c(t)M(x, Sy, Sx, (2 - \alpha)t) \\
+ e(t)M(x, y, y, t)
\]
for all \(x, y \in X\) and \(\alpha \in (0, 2)\), where \(a, b, c, d, e : [0, \infty) \rightarrow [0, 1]\) are five functions such that
\[
a(t) + b(t) + c(t) + d(t) + e(t) = 1, \quad \forall t \in [0, \infty).
\]
Then \(S\) have a unique common fixed point in \(X\).

Corollary 2.7. Let \((X, M, \ast)\) be a complete \(M\)-fuzzy metric space with \(t \ast t = t\) for all \(t \in [0, 1]\). Let \(S\) be a mapping from \(X\) into itself such that there exists \(k \in (0,1)\) such that
\[
M(Sx, y, y, kt) \\
\geq a(t)M(x, Sx, Sx, t) + b(t)M(x, y, y, \alpha t) \\
+ c(t)M(y, Sx, Sx, (2 - \alpha)t) + d(t)M(x, y, y, t)
\]
for all \(x, y \in X\) and \(\alpha \in (0, 2)\), where \(a, b, c, d : [0, \infty) \rightarrow [0, 1]\) are five functions such that
\[
a(t) + b(t) + c(t) + d(t) = 1, \quad \forall t \in [0, \infty).
\]
Then \(S\) have a unique common fixed point in \(X\).

Corollary 2.8. Let \((X, M, \ast)\) be a complete \(M\)-fuzzy metric space with \(t \ast t = t\) for all \(t \in [0, 1]\). Let \(S\) and \(T\) be mappings from \(X\) into itself such that there exists \(k \in (0,1)\) such that
\[
M(S^n x, T^m y, T^m y, kt) \\
\geq a(t)M(x, S^n x, S^n x, t) + b(t)M(y, T^m y, T^m y, t) \\
+ c(t)M(x, T^m y, T^m y, (2 - \alpha)t) + d(t)M(y, S^n x, S^n x, (2 - \alpha)t) \\
+ e(t)M(x, y, y, t)
\]
for all \(x, y \in X\), \(\alpha \in (0, 2)\) and \(n, m \geq 2\), where \(a, b, c, d, e : [0, \infty) \rightarrow [0, 1]\) are five functions such that
\[
a(t) + b(t) + c(t) + d(t) + e(t) = 1, \quad \forall t \in [0, \infty).
\]
If \(S^n T = T S^n\) and \(T^m S = S T^m\), then \(S\) and \(T\) have a unique common fixed point in \(X\).
Proof. By Theorem 2.4, $S^n$ and $T^m$ have a unique common fixed point in $X$. That is, there exists a unique point $z \in X$ such that $S^n(z) = T^m(z) = z$. Since $S(z) = S(S^n(z)) = S^n(S(z))$ and $T(z) = T(T^m(z)) = T^m(T(z))$, that is, $S(z)$ is fixed point $S^n$ and $T(z)$ and so $S(z) = z$. Similarly, $T(z) = z$. This completes the proof. \hfill \Box

Corollary 2.9. Let $(X, \mathcal{M}, *)$ be a complete $\mathcal{M}$-fuzzy metric space with $t * t = t$ for all $t \in [0, 1]$. Let $S$ and $T$ be mappings from $X$ into itself such that there exists $k \in (0, 1)$ such that

$$M(Sx, Ty, Ty, kt) \geq a(t)M(x, Sx, Sx, t) + b(t)M(y, Ty, Ty, t)$$

for all $x, y \in X$ and $\alpha \in (0, 2)$, where $a, b : [0, \infty) \rightarrow [0, 1]$ are two functions such that

$$a(t) + b(t) = 1, \quad \forall t \in [0, \infty).$$

Then $S$ and $T$ have a unique common fixed point in $X$. 

References


